Formal definition of a system of push down automata

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We define a transducer pushdown automaton (TPA) to be a finite automaton that contains an unbounded stack and an unbounded output tape in addition to its finite input tape. More formally, a TPA $M_0$ is a tuple $M_0 = (K_0, \Sigma_0, \Gamma_0, \Delta_0, s_0, F_0)$ where $K_0$ is a finite set of states, $\Sigma_0$ and $\Gamma_0$ are finite sets of symbols (the input and output alphabet, and the stack alphabet, respectively), $\Delta_0$ is the transition function, $s_0 \in K_0$ is the start state, and $F_0 \subseteq K_0$ is the set of final states. The transition function $\Delta_0$ is of the form $\Delta_0 \subseteq (K_0 \times \Sigma_0 \times (\Gamma_0 \cup \{\varepsilon\})) \rightarrow (K_0 \times (\Gamma_0 \cup \{\varepsilon\}) \times \Sigma_0^*)$, where $\varepsilon$ is the empty word. Each transition consists of $M_0$ reading exactly one input symbol, popping at most one symbol from the stack, changing state, pushing at most one symbol to the stack, and appending a string to the output tape. Each configuration of $M_0$ is an element of $K_0 \times \Sigma_0^\ast \times \Gamma_0^\ast \times \Sigma_0^\ast$, consisting of the current state, the remaining symbols on the input tape, the contents of the stack, and the contents of the output tape. A transition exists between configuration $C = (\alpha, aw, bs, o)$ and configuration $C' = (\beta, w, cs, od)$ if there exists a rule $(\alpha, a, b) \rightarrow (\beta, c, d) \in \Delta_0$, where $\alpha, \beta \in K_0$, $a \in \Sigma_0$, $w \in \Sigma_0^\ast$, $b, c \in (\Gamma_0 \cup \{\varepsilon\})$, $s \in \Gamma_0^\ast$, $o, d \in \Sigma_0^\ast$. It can be seen that each TPA is deterministic and always halts.

We define $M = (K, \Sigma, \Gamma, \Delta, S, F)$ as a system of $n$ TPAs, where each TPA $i$ has a unique finite set of $K_i$ states such that

$$K = \bigcup_{i=0}^{n-1} K_i, \bigcap_{i=0}^{n-1} K_i = \emptyset,$$

$\Sigma$ is a finite alphabet of input/output symbols, $\# \not\in \Sigma$, $\Gamma$ is a finite alphabet of stack symbols, $\# \not\in \Gamma$, $S = \{s_i : s_i \in K_i, 0 \leq i < n\}$ contains the start state for each TPA $i$, and $F$ contains the final states for each TPA $i$ such that

$$F = \bigcup_{i=0}^{n-1} F_i, F_i \subseteq K_i.$$

The transition function, $\Delta \subseteq \Delta' \cup \Delta''$, is defined as a subset of the union of the sets of all intra-TPA transitions and inter-TPA transitions. The set of all
The set of all intra-TPA transitions is
\[
\Delta' = \bigcup_{i=0}^{n-1} ((K_i \times \Sigma \times (\Gamma \cup \{\varepsilon\})) \rightarrow (K_i \times (\Gamma \cup \{\varepsilon\}) \times \Sigma^*)).
\]

The set of all inter-TPA transitions is
\[
\Delta'' = \bigcup_{i=0}^{n-2} ((F_i \times \{\#\} \times \{\varepsilon\}) \rightarrow (\{s_{i+1}\} \times \{\varepsilon\} \times \{\#\})).
\]

A configuration of $M$ is an element of $K \times (\Sigma^* \times \Gamma^* \times \Sigma^*)^n$ where $\Sigma^* = (\Sigma^* \cup \{\#\})$. The initial configuration of $M$ is
\[
(s_0, (w\#, \varepsilon, \varepsilon), \varepsilon, \varepsilon, \varepsilon),...,(\varepsilon, \varepsilon, \varepsilon),
\]
where $w \in \Sigma^*$ is the input to $M$. A final configuration of $M$ is of the form
\[
(f, (\varepsilon, \gamma_0, \varepsilon), (\varepsilon, \gamma_1, \varepsilon),...,(\#, \gamma_{n-1}, r)),
\]
where $f \in F_{n-1}$, $r \in \Sigma^*$, and $\gamma_i \in \Gamma^*$, $0 \leq i < n$.

Let `$\vdash$' be a binary relation on configurations called the transition. A transition exists between configuration $C_1$ and configuration $C_2$, denoted $C_1 \vdash C_2$, if $C_1$ is of the form
\[
(\alpha, (\varphi, (aw\#, bs, o), \chi)),
\]
where $\alpha \in K_i$, $w, a, o \in \Sigma^*$, $s \in \Gamma^*$, $b \in (\Gamma \cup \{\varepsilon\})$, $\varphi \in (\Sigma^* \times \Gamma^* \times \Sigma^*)^p$, $\chi \in (\Sigma^* \times \Gamma^* \times \Sigma^*)^q$, $p + q + 1 = n$ and $C_2$ is of the form
\[
(\beta, (\varphi, (w\#, cs, od), \chi)),
\]
where $\beta \in K_i$, $d \in \Sigma^*$, $c \in (\Gamma \cup \{\varepsilon\})$ and a transition rule of the following form exists in $\Delta$
\[
(\alpha, a, b) \rightarrow (\beta, c, d),
\]
or, if $C_1$ is of the form
\[
(f, (\psi, (\#, \gamma_{p+1}, r), (\varepsilon, \varepsilon, \varepsilon), \omega)),
\]
where $f \in F_i$, $\psi \in (\Sigma^* \times \Gamma^* \times \Sigma^*)^p$, $\omega \in (\Sigma^* \times \Gamma^* \times \Sigma^*)^q$, $p + q + 2 = n$, $\gamma_{p+1} \in \Gamma^*$, $r \in \Sigma^*$ and $C_2$ is of the form
\[
(s_{i+1}, (\psi(\varepsilon, \gamma_{p+1}, \varepsilon), (r\#, \varepsilon, \varepsilon) \omega)),
\]
where a transition rule of the following form exists in $\Delta$
\[
(f, \#, \varepsilon) \rightarrow (s_{i+1}, \varepsilon, \#)).
\]

We denote the reflexive and transitive closure of $\vdash$ as $\vdash^*$. An accepting computation for $M$ with input $w$ exists if and only if $C_{initial} \vdash^* C_{final}$ where $C_{initial}$ is an initial configuration and $C_{final}$ is a final configuration.